

Line k -Arboricity in Product Networks ^{*}

Yaping Mao^{1,2}, Zhiwei Guo^{1,†}, Nan Jia¹, He Li¹

¹Department of Mathematics, Qinghai Normal University,
Xining, Qinghai 810008, China

²Key Laboratory of IOT of Qinghai Province,
Xining, Qinghai 810008, China

E-mails: maoyaping@ymail.com; guozhiweic@yahoo.com
jianan66ss@yahoo.com; lihe0520@yahoo.com

Abstract

A *linear k -forest* is a forest whose components are paths of length at most k . The *linear k -arboricity* of a graph G , denoted by $\text{la}_k(G)$, is the least number of linear k -forests needed to decompose G . Recently, Zuo, He and Xue studied the exact values of the linear $(n-1)$ -arboricity of Cartesian products of various combinations of complete graphs, cycles, complete multipartite graphs. In this paper, for general k we show that $\max\{\text{la}_k(G), \text{la}_\ell(H)\} \leq \text{la}_{\max\{k,\ell\}}(G \square H) \leq \text{la}_k(G) + \text{la}_\ell(H)$ for any two graphs G and H . Denote by $G \circ H$, $G \times H$ and $G \boxtimes H$ the lexicographic product, direct product and strong product of two graphs G and H , respectively. We also derive upper and lower bounds of $\text{la}_k(G \circ H)$, $\text{la}_k(G \times H)$ and $\text{la}_k(G \boxtimes H)$ in this paper. The linear k -arboricity of a 2-dimensional grid graph, a r -dimensional mesh, a r -dimensional torus, a r -dimensional generalized hypercube and a 2-dimensional hyper Petersen network are also studied.

Keywords: Linear k -forest, linear k -arboricity, Cartesian product, complete product, lexicographical product, strong product, direct product.

AMS subject classification 2010: 05C15, 05C76, 05C78.

1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to the book [6] for graph theoretical notation and terminology not described here. Let N be the

^{*}Supported by the National Science Foundation of China (Nos. 11551001, 11161037, 11461054) and the Science Found of Qinghai Province (No. 2014-ZJ-907).

[†]Corresponding author

set of natural numbers and let $[a, b]$ be the set $\{n \in N \mid a \leq n \leq b\}$. A *decomposition* of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. If a graph G has a decomposition G_1, G_2, \dots, G_t , then we say that G_1, G_2, \dots, G_t *decompose* G or G *can be decomposed into* G_1, G_2, \dots, G_t . Furthermore, a *linear k -forest* is a forest whose components are paths of length at most k . The *linear k -arboricity* of a graph G , denoted by $\text{la}_k(G)$, is the least number of linear k -forests needed to decompose G .

The notion of linear k -arboricity of a graph was first introduced by Habib and Peroche [16], which is a natural generalization of edge-coloring. Clearly, a linear 1-forest is induced by a matching, and $\text{la}_1(G)$ is the chromatic index $\chi'(G)$ of a graph G . Moreover, the linear k -arboricity $\text{la}_k(G)$ is also a refinement of the ordinary linear arboricity $\text{la}(G)$ (or $\text{la}_\infty(G)$) which is the case when every component of each forest is a path with no length constraint. By the way, the notion of linear arboricity was introduced earlier by Harary in [17]. For more details on linear k -arboricity, we refer to [2, 3, 4, 8, 9, 10].

In graph theory, Cartesian product, strong product, lexicographical product and direct product are four of main products, each with its own set of applications and theoretical interpretations. Product networks were proposed based upon the idea of using the cross product as a tool for “combining” two known graphs with established properties to obtain a new one that inherits properties from both [12]. Recently, there has been an increasing interest in a class of interconnection networks called Cartesian product networks; see [1, 12, 21].

The join, Cartesian, lexicographical, strong and direct products are defined as follows.

- The *join* or *complete product* $G \vee H$ of two disjoint graphs G and H , is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$.
- The *Cartesian product* $G \square H$ of two graphs G and H , is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $(v, v') \in E(H)$, or $v = v'$ and $(u, u') \in E(G)$.
- The *lexicographic product* $G \circ H$ of graphs G and H has the vertex set $V(G \circ H) = V(G) \times V(H)$, and two vertices $(u, v), (u', v')$ are adjacent if $uu' \in E(G)$, or if $u = u'$ and $vv' \in E(H)$.
- The *strong product* $G \boxtimes H$ of graphs G and H has the vertex set $V(G) \times V(H)$. Two vertices (u, v) and (u', v') are adjacent whenever $uu' \in E(G)$ and $v = v'$, or $u = u'$ and $vv' \in E(H)$, or $uu' \in E(G)$ and $vv' \in E(H)$.
- The *direct product* $G \times H$ of graphs G and H has the vertex set $V(G) \times V(H)$. Two vertices (u, v) and (u', v') are adjacent if the projections on both coordinates are adjacent, i.e., $uu' \in E(G)$ and $vv' \in E(H)$.

From the definition of linear k -arboricity and the structure of graph product, the following result is immediate.

Observation 1.1 *Let G and H be two graphs. Then*

$$\text{la}_k(G \boxtimes H) \leq \text{la}_k(G \square H) + \text{la}_k(G \times H).$$

Xue and Zuo [26] investigated the linear $(n - 1)$ -arboricity of complete multipartite graphs. Recently, Zuo, He and Xue [29] studied the exact values of the linear $(n - 1)$ -arboricity of Cartesian products of various combinations of complete graphs, cycles, complete multipartite graphs.

In this paper, we consider four standard products: the lexicographic, the strong, the Cartesian and the direct with respect to the linear k -arboricity. Every of these four products will be treated in one of the forthcoming subsections in Section 2. In Section 3, we demonstrate the usefulness of the proposed constructions by applying them to some instances of product networks.

2 Results for general graphs

As usual, the *union* of two graphs G and H is the graph, denoted by $G \cup H$, with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The disjoint union of k copies of the same graph G is denoted by kG . The *join* $G \vee H$ of two disjoint graphs G and H is obtained from $G \cup H$ by joining each vertex of G to every vertex of H . In the sequel, let $K_{s,t}$, C_n , K_n and P_n denote the complete bipartite graph of order $s + t$ with part sizes s and t , cycle of order n , complete graph of order n , and path of order n , respectively.

In the sequel, let G and H be two connected graphs with $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(H) = \{v_1, v_2, \dots, v_m\}$, respectively. Then $V(G * H) = \{(u_i, v_j) \mid 1 \leq i \leq n, 1 \leq j \leq m\}$, where $*$ denotes a kind of graph product operations. For $v \in V(H)$, we use $G(v)$ to denote the subgraph of $G * H$ induced by the vertex set $\{(u_i, v) \mid 1 \leq i \leq n\}$. Similarly, for $u \in V(G)$, we use $H(u)$ to denote the subgraph of $G * H$ induced by the vertex set $\{(u, v_j) \mid 1 \leq j \leq m\}$.

The following observations are immediate.

Observation 2.1 *Let H be a subgraph of G . If $\ell \geq k$, then*

$$\text{la}_\ell(H) \leq \text{la}_k(G).$$

Observation 2.2 [27] *If a graph G is the edge-disjoint union of two subgraphs G_1 and G_2 , then*

$$\text{la}_k(G) \leq \text{la}_k(G_1) + \text{la}_k(G_2).$$

Observation 2.3 [27] *If a graph G is the disjoint union of two subgraphs G_1 and G_2 , then*

$$\text{la}_k(G) = \max\{\text{la}_k(G_1), \text{la}_k(G_2)\}.$$

Observation 2.4 *If G is not a forest, then $\text{la}_k(G) \geq 2$ for $k \geq 1$.*

Lemma 2.1 [10] *For a graph G of order n ,*

$$\Delta(G) + 1 \geq \chi'(G) = \text{la}_1(G) \geq \text{la}_2(G) \geq \dots \geq \text{la}_{n-1}(G) = \text{la}(G),$$

where $\chi'(G)$ denotes the edge chromatic number of G .

Lemma 2.2 [10] *For a graph G ,*

$$\text{la}_k(G) \geq \max \left\{ \left\lceil \frac{\Delta(G)}{2} \right\rceil, \left\lceil \frac{|E(G)|}{\lfloor \frac{k|V(G)|}{k+1} \rfloor} \right\rceil \right\}.$$

2.1 For Cartesian product

We first give the bounds for general graphs.

Theorem 2.5 *Let G and H be two graphs. Then*

$$\max\{\text{la}_k(G), \text{la}_\ell(H)\} \leq \text{la}_{\max\{k,\ell\}}(G \square H) \leq \text{la}_k(G) + \text{la}_\ell(H).$$

Moreover, the upper bound is sharp.

Proof. Set $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(H) = \{v_1, v_2, \dots, v_m\}$. Let $\text{la}_k(G) = p$ and $\text{la}_\ell(H) = q$. Since $\text{la}_k(G) = p$, it follows that there are p linear k -forests in G . Then, since $\text{la}_k(G(v_i)) = p$, it follows that there are p linear k -forests in $G(v_i)$, say $F_{i,1}, F_{i,2}, \dots, F_{i,p}$. Similarly, since $\text{la}_\ell(H) = q$, it follows that there are q linear ℓ -forests in H . Then, since $\text{la}_\ell(H(u_i)) = q$, it follows that there are q linear ℓ -forests in $H(u_i)$, say $F'_{i,1}, F'_{i,2}, \dots, F'_{i,q}$. Set $F_j = \bigcup_{i=1}^m F_{i,j}$ where $1 \leq j \leq p$, and $F'_j = \bigcup_{i=1}^n F'_{i,j}$ where $1 \leq j \leq q$. Clearly, $F_1, F_2, \dots, F_p, F'_1, F'_2, \dots, F'_q$ are $p + q$ linear k -forest in $G \square H$. So $\text{la}_{\max\{k,\ell\}}(G \square H) \leq \text{la}_k(G) + \text{la}_\ell(H)$. From Observation 2.1, we have $\text{la}_{\max\{k,\ell\}}(G \square H) \geq \max\{\text{la}_k(G), \text{la}_\ell(H)\}$. ■

The following corollary is a generalization of the above result.

Corollary 2.6 *Let G_1, G_2, \dots, G_r be graphs. Then*

$$\begin{aligned} \max\{\text{la}_{k_1}(G_1), \text{la}_{k_2}(G_2), \dots, \text{la}_{k_r}(G_r)\} &\leq \text{la}_{\max\{k_1, k_2, \dots, k_r\}}(G_1 \square G_2 \square \dots \square G_r) \\ &\leq \text{la}_{k_1}(G_1) + \text{la}_{k_2}(G_2) + \dots + \text{la}_{k_r}(G_r). \end{aligned}$$

Moreover, the bounds are sharp.

2.2 For complete product

The following results were obtained by Dirac [13]; see Laskar and Auerbach [22].

Proposition 2.7 [13, 22] (1) *For all even $r \geq 2$, $K_{r,r}$ is the union of its $\frac{1}{2}r$ Hamiltonian cycles.*

(2) *For all odd $r \geq 3$, $K_{r,r}$ is the union of its $\frac{1}{2}r$ Hamiltonian cycles and one perfect matching.*

For complete product, we have the following.

Theorem 2.8 *Let G and H be two graphs. Then*

$$\max \left\{ \left\lceil \frac{\Delta(G) + |V(H)|}{2} \right\rceil, \left\lceil \frac{\Delta(H) + |V(G)|}{2} \right\rceil \right\} \leq \text{la}_k(G \vee H) \leq \text{la}_k(G) + \text{la}_k(H) + |V(H)|.$$

Proof. Set $|V(G)| = n$ and $|V(H)| = m$. Without loss of generality, let $n \leq m$. Let $G' = G \cup (m - n)K_1$. Then $|V(G')| = m$ and $G \vee H$ is a subgraph of $G' \vee H$. Since $\text{la}_k(G) = p$, it follows that there are p linear k -forests in G , say F_1, F_2, \dots, F_p . From the structure of G' , F_1, F_2, \dots, F_p are linear k -forests in G' , and hence $\text{la}_k(G') \leq p$. From Observation 2.1, we have $p = \text{la}_k(G) \leq \text{la}_k(G') \leq p$, and hence $\text{la}_k(G') = p$. Since $\text{la}_k(H) = q$, it follows that there are q linear k -forests in H , say F'_1, F'_2, \dots, F'_q . Note that the subgraph induced by all the vertices of $G' \vee H$ is a complete bipartite graph $K_{m,m}$. From Proposition 2.7, $K_{m,m}$ can be decomposed into m perfect matchings, say M_1, M_2, \dots, M_m . These perfect matchings are m linear 1-forests in $G' \vee H$. Observe that

$$E(G' \vee H) = \left(\bigcup_{i=1}^m M_i \right) \cup \left(\bigcup_{i=1}^p E(F_i) \right) \cup \left(\bigcup_{i=1}^q E(F'_i) \right).$$

Then $F_1, F_2, \dots, F_p, F'_1, F'_2, \dots, F'_q, M_1, M_2, \dots, M_m$ form $p + q + m$ linear k -forests in $G' \vee H$. So $\text{la}_k(G \vee H) \leq \text{la}_k(G' \vee H) \leq \text{la}_k(G) + \text{la}_k(H) + |V(H)|$. Note that $\Delta(G \vee H) \geq \max\{\Delta(G) + |V(H)|, \Delta(H) + |V(G)|\}$. From Lemma 2.2, we have

$$\text{la}_k(G \vee H) \geq \max \left\{ \left\lceil \frac{\Delta(G) + |V(H)|}{2} \right\rceil, \left\lceil \frac{\Delta(H) + |V(G)|}{2} \right\rceil \right\},$$

as desired. ■

2.3 For lexicographical product

From the definition, the lexicographic product graph $G \circ H$ is a graph obtained by replacing each vertex of G by a copy of H and replacing each edge of G by a complete bipartite graph $K_{m,m}$. For an edge $e = u_i u_j \in E(G)$ ($1 \leq i, j \leq n$), the induced subgraph obtained from the edges between the vertex set $V(H(u_i)) = \{(u_i, v_1), (u_i, v_2), \dots, (u_i, v_m)\}$

and the vertex set $V(H(u_j)) = \{(u_j, v_1), (u_j, v_2), \dots, (u_j, v_m)\}$ in $G \circ H$ is a complete equipartition bipartite graph of order $2m$, denoted by K_e or K_{u_i, u_j} .

From Proposition 2.7, K_e can be decomposed into m perfect matching, denoted by $M_1^e, M_2^e, \dots, M_m^e$. We now in a position to give the result for lexicographical product.

Theorem 2.9 *Let G and H be two graphs. Then*

$$\left\lceil \frac{\Delta(H) + |V(H)|\Delta(G)}{2} \right\rceil \leq \text{la}_{\max\{k, \ell\}}(G \circ H) \leq \text{la}_k(G)|V(H)| + \text{la}_\ell(H).$$

Moreover, the bounds are sharp.

Proof. Set $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(H) = \{v_1, v_2, \dots, v_m\}$. For $k \leq \ell$, we need to show that $\text{la}_\ell(G \circ H) \leq \text{la}_k(G)|V(H)| + \text{la}_\ell(H)$. Let $\text{la}_k(G) = p$ and $\text{la}_\ell(H) = q$. Since $\text{la}_\ell(H) = q$, it follows that there are q linear ℓ -forest in H . Then, since $\text{la}_\ell(H(u_i)) = q$, it follows that there are q linear ℓ -forests in $H(u_i)$, say $F'_{i,1}, F'_{i,2}, \dots, F'_{i,q}$. Set $F'_j = \bigcup_{i=1}^q F'_{i,j}$ where $1 \leq j \leq q$.

Since $\text{la}_k(G) = p$, it follows that there are p linear k -forests in G , say F_1, F_2, \dots, F_p . For each linear k -forest F_i ($1 \leq i \leq p$) in G , we define a subgraph \mathcal{F}_i of $G \circ H$ corresponding to F_i as follows: $V(\mathcal{F}_i) = V(F_i \circ H)$ and $E(\mathcal{F}_i) = \{(u_p, v_s)(u_q, v_t) \mid u_p u_q \in E(F_i), u_p, u_q \in V(G), v_s, v_t \in V(H)\}$. We call \mathcal{F}_i a *blow-up linear k -forest corresponding to T_i in G* ; see Figure 1 for an example. For each i ($1 \leq i \leq p$) and each j ($1 \leq j \leq m$), we define another

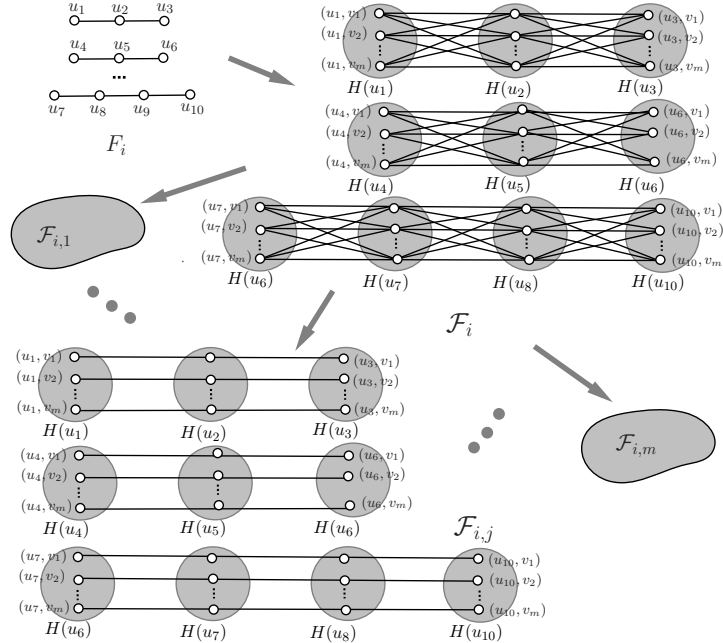


Figure 1: The blow-up linear k -forest \mathcal{F}_i and parallel linear k -forest $\mathcal{F}_{i,j}$ in $G \circ H$ corresponding to F_i in G .

subgraph $\mathcal{F}_{i,j}$ of $G \circ H$ corresponding to T_i in G as follows: $V(\mathcal{F}_{i,j}) = V(F_i \circ H)$ and $E(\mathcal{F}_{i,j}) = \bigcup_{e \in E(F_i)} M_{i,j}^e$, where $M_{i,j}^e$ is a matching of K_e . We call $\mathcal{F}_{i,j}$ a *parallel linear k -forest of $G \circ H$ corresponding to the forest F_i in G* ; see Figure 1 for an example. Note that all the parallel linear k -forests in $\{\mathcal{F}_{i,j} \mid 1 \leq i \leq p, 1 \leq j \leq m\}$ are pm linear k -forests of $G \circ H$.

Since $k \leq \ell$, it follows that the forests F'_1, F'_2, \dots, F'_q and all the forests in $\{\mathcal{F}_{i,j} \mid 1 \leq i \leq p, 1 \leq j \leq m\}$ form $pm + q$ linear ℓ -forests of $G \circ H$. Observe that each edge of $G \circ H$ belongs to one of the above linear ℓ -forests. So $\text{la}_\ell(G \circ H) \leq \text{la}_k(G)|V(H)| + \text{la}_\ell(H)$.

For $\ell \leq k$, we can prove that $\text{la}_k(G \circ H) \leq \text{la}_k(G)|V(H)| + \text{la}_\ell(H)$ similarly. We conclude that $\text{la}_{\max\{k,\ell\}}(G \circ H) \leq \text{la}_k(G)|V(H)| + \text{la}_\ell(H)$.

Note that $\Delta(G \circ H) \geq \Delta(H) + |V(H)|\Delta(G)$. From Lemma 2.2, we have

$$\text{la}_{\max\{k,\ell\}}(G \circ H) \geq \left\lceil \frac{\Delta(H) + |V(H)|\Delta(G)}{2} \right\rceil,$$

as desired. ■

2.4 For direct product

For direct product, we have the following.

Theorem 2.10 *Let G and H be two graphs. Then*

$$\left\lceil \frac{\Delta(G)\Delta(H)}{2} \right\rceil \leq \text{la}_{\max\{k,\ell\}}(G \times H) \leq 2\text{la}_k(G)\text{la}_\ell(H).$$

Moreover, the bounds are sharp.

Proof. From Lemma 2.2, we have

$$\text{la}_{\max\{k,\ell\}}(G \times H) \geq \left\lceil \frac{\Delta(G \times H)}{2} \right\rceil \geq \left\lceil \frac{\Delta(G)\Delta(H)}{2} \right\rceil.$$

It suffices to show that $\text{la}_{\max\{k,\ell\}}(G \times H) \leq 2\text{la}_k(G)\text{la}_\ell(H)$.

We now give the proof of this theorem, with a running example (corresponding to Figure 2). From the symmetry of direct product, we can assume $k \leq \ell$. We only need to show $\text{la}_\ell(G \times H) \leq 2\text{la}_k(G)\text{la}_\ell(H)$. Set $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(H) = \{v_1, v_2, \dots, v_m\}$. Let $\text{la}_k(G) = p$ and $\text{la}_\ell(H) = q$. Since $\text{la}_k(G) = p$, it follows that there are p linear k -forests in G , say F_1, F_2, \dots, F_p . For each F_i ($1 \leq i \leq p$), we assume that $P_{i,1}, P_{i,2}, \dots, P_{i,x}$ are all the paths in F_i . Then $F_i = \bigcup_{j=1}^x P_{i,j}$ where $1 \leq i \leq p$. Take for example, let $P_{i,1} = P_2$, $P_{i,2} = P_3$, $P_{i,3} = P_4$; see Figure 2 (a). Then $F_i = P_{i,1} \cup P_{i,2} \cup P_{i,3}$.

For each $P_{i,j}$, we let $P_{i,j} = u_1^{i,j} u_2^{i,j} \dots u_a^{i,j}$, where $1 \leq j \leq x$. We first define two subgraphs $P_{i,j}^1, P_{i,j}^2$ induced by the edges in

$$\begin{aligned} E(P_{i,j}^1) &= \{u_{2r-1}^{i,j} u_{2r}^{i,j} \mid 1 \leq r \leq \lfloor a/2 \rfloor\}, \\ E(P_{i,j}^2) &= \{u_{2r}^{i,j} u_{2r+1}^{i,j} \mid 1 \leq r \leq \lfloor a/2 \rfloor\}, \end{aligned}$$

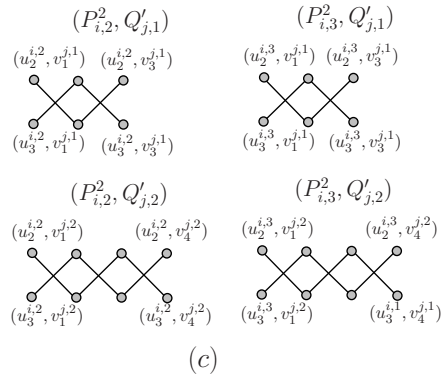
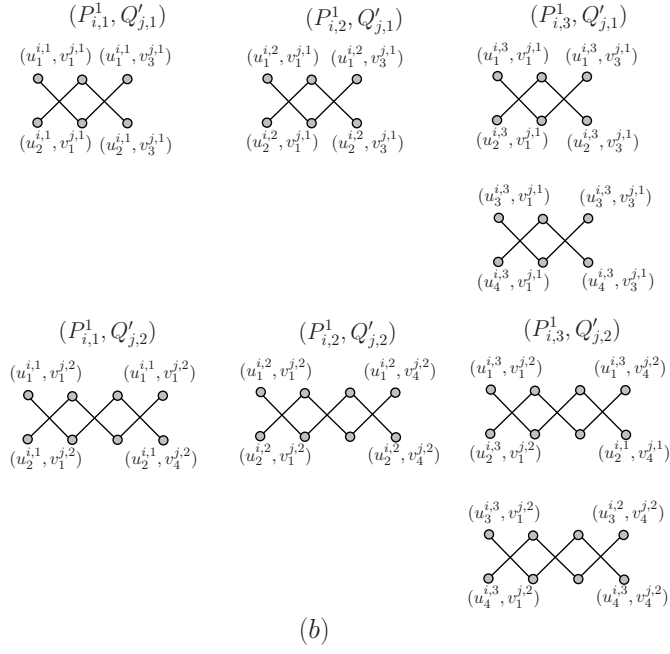
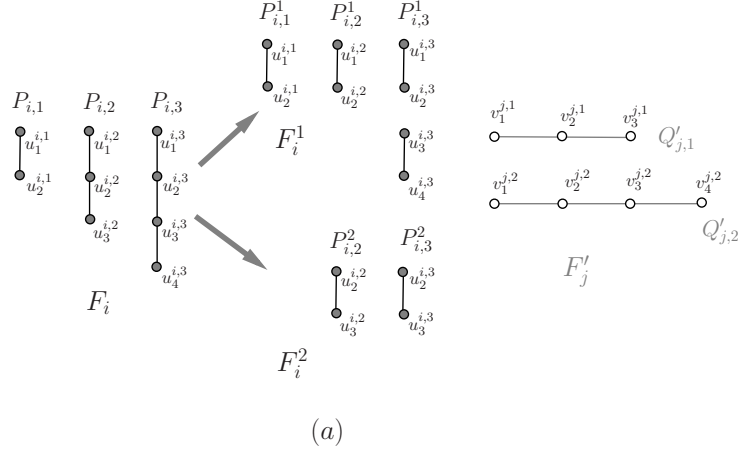


Figure 2: The running example for Theorem 2.10.

respectively. Next, we set $F_i^1 = \bigcup_{j=1}^x P_{i,j}^1$ and $F_i^2 = \bigcup_{j=1}^x P_{i,j}^2$. For the above example, we have $P_{i,1} = u_1^{i,1} u_2^{i,1}$, $P_{i,2} = u_1^{i,2} u_2^{i,2} u_3^{i,2}$ and $P_{i,3} = u_1^{i,3} u_2^{i,3} u_3^{i,3} u_4^{i,3}$. Then $P_{i,1}^1$ is the subgraph induced by the edge $u_1^{i,1} u_2^{i,1}$, $P_{i,2}^1$ is the subgraph induced by the edge $u_1^{i,2} u_2^{i,2}$, and $P_{i,3}^1$ is the subgraph induced by the edges in $\{u_1^{i,3} u_2^{i,3}, u_3^{i,3} u_4^{i,3}\}$. Furthermore, $P_{i,2}^2$ is the subgraph induced by the edge $u_2^{i,2} u_3^{i,2}$ and $P_{i,3}^2$ is the subgraph induced by the edge $u_2^{i,3} u_3^{i,3}$; see Figure 2 (a). Note that $F_i^1 = \bigcup_{j=1}^3 P_{i,j}^1$ and $F_i^2 = \bigcup_{j=2}^3 P_{i,j}^2$.

Since $\text{la}_k(H) = q$, it follows that there are q linear k -forests in H , say F'_1, F'_2, \dots, F'_q . For each F'_j ($1 \leq j \leq q$), we assume that $Q'_{j,1}, Q'_{j,2}, \dots, Q'_{j,y}$ are all the paths in F'_j . Then $F'_j = \bigcup_{i=1}^y Q'_{j,i}$ where $1 \leq j \leq q$. Set $Q'_{j,i} = v_{j,i}^1 v_{j,i}^2 \dots v_{j,i}^b$, where $1 \leq i \leq y$. For the above example, we have $Q'_{j,1} = v_1^{j,1} v_2^{j,1} v_3^{j,1}$ and $Q'_{j,2} = v_1^{j,2} v_2^{j,2} v_3^{j,2} v_4^{j,2}$. Then $F'_j = Q'_{j,1} \cup Q'_{j,2}$.

We now decompose $G \times H$ into $2pq$ linear ℓ -forests such that each of them is formed from the paths in F_i^1 or F_i^2 and the paths in F'_j , where $1 \leq i \leq p$ and $1 \leq j \leq q$. Note that the subgraph $F_{i,j}^*$ induced by the edges in

$$\{(u_1, v_1)(u_2, v_2), (u_1, v_2)(u_2, v_1) \mid v_1 v_2 \in E(F'_j), u_1 u_2 \in E(F_i^1)\}$$

and the subgraph $F_{i,j}^{**}$ induced by the edges in

$$\{(u_1, v_1)(u_2, v_2), (u_1, v_2)(u_2, v_1) \mid v_1 v_2 \in E(F'_j), u_1 u_2 \in E(F_i^2)\}$$

are $2pq$ linear ℓ -forests in $G \times H$, where $1 \leq i \leq p$ and $1 \leq j \leq q$. Note that each edge of $G \times H$ belongs to one of the above linear ℓ -forests. So $\text{la}_{\max\{k,\ell\}}(G \times H) = \text{la}_\ell(G \times H) \leq 2\text{la}_k(G)\text{la}_\ell(H)$. \blacksquare

2.5 For strong product

For direct product, we have the following.

Theorem 2.11 *Let G and H be two graphs. Then*

$$\left\lceil \frac{\Delta(G)\Delta(H) + \Delta(G) + \Delta(H)}{2} \right\rceil \leq \text{la}_{\max\{k,\ell\}}(G \boxtimes H) \leq \text{la}_k(G) + \text{la}_\ell(H) + 2\text{la}_k(G)\text{la}_\ell(H).$$

Moreover, the bounds are sharp.

Proof. Note that $\Delta(G \boxtimes H) \geq \Delta(G)\Delta(H) + \Delta(G) + \Delta(H)$. From Lemma 2.2, we have

$$\text{la}_{\max\{k,\ell\}}(G \boxtimes H) \geq \left\lceil \frac{\Delta(G)\Delta(H) + \Delta(G) + \Delta(H)}{2} \right\rceil.$$

Since $E(G \boxtimes H) = E(G \times H) \cup E(G \square H)$, it follows from Observation 2.2 that

$$\begin{aligned} \text{la}_{\max\{k,\ell\}}(G \boxtimes H) &\leq \text{la}_{\max\{k,\ell\}}(G \square H) + \text{la}_{\max\{k,\ell\}}(G \times H) \\ &\leq \text{la}_k(G) + \text{la}_\ell(H) + 2\text{la}_k(G)\text{la}_\ell(H), \end{aligned}$$

as desired. \blacksquare

3 Results for product networks

In this section, we demonstrate the usefulness of the proposed constructions by applying them to some instances of Cartesian product networks. We first study the linear k -arboricity of a path, a cycle, a complete graph and a Peterson graph.

Lemma 3.1 *For path P_n ($n \geq 2$),*

$$\begin{cases} \text{la}_k(P_n) = 1 & \text{if } k \geq n - 1 \\ \text{la}_k(P_n) = 2 & \text{if } 1 \leq k < n - 1. \end{cases}$$

Proof. If $k \geq n - 1$, then P_n itself is a line k -forest, and hence $\text{la}_k(P_n) = 1$. Suppose $k < n - 1$. Let M be a maximum matching of P_n . Then $P_n \setminus M$ contains a matching, say M' . For $1 \leq k < n - 1$, M, M' are two line k -forests, and hence $\text{la}_k(P_n) \leq 2$. From Observation 2.4, we have $\text{la}_k(P_n) \geq 2$. So $\text{la}_k(P_n) = 2$ for $1 \leq k < n - 1$. ■

Lemma 3.2 *For cycle C_n ($n \geq 3$),*

$$\begin{cases} \text{la}_k(C_n) = 2 & \text{if } n \text{ is even, } k \geq 1 \\ \text{la}_k(C_n) = 3 & \text{if } n \text{ is odd, } k = 1 \\ \text{la}_k(C_n) = 2 & \text{if } n \text{ is odd, } k \geq 2. \end{cases}$$

Proof. Suppose that n is even and $k \geq 1$. Since n is even, it follows that there exists a perfect matching of G , say M . Then $M' = E(C_n) \setminus M$ is also a perfect matching of G . Clearly, M and M' are two line k -forests, and hence $\text{la}_k(C_n) \leq 2$. From Observation 2.4, we have $\text{la}_k(C_n) = 2$ for n is even and $k \geq 1$.

Suppose that n is odd and $k \geq 2$. Since n is odd, it follows that there exists a maximum matching of size $\frac{n-1}{2}$, say M . Set $M' = E(C_n) \setminus M$. Clearly, M and M' are two line k -forests, and hence $\text{la}_k(C_n) \leq 2$. From Observation 2.4, we have $\text{la}_k(C_n) = 2$ for n is even and $k \geq 1$.

Suppose that n is odd and $k = 1$. Set $V(C_n) = \{v_1, v_2, \dots, v_n\}$. We divide the edge set of G into three categories: $M_1 = \{v_{2i-1}v_{2i} \mid 1 \leq i \leq r\}$, $M_2 = \{v_{2i}v_{2i+2} \mid 1 \leq i \leq r\}$ and $M_3 = \{v_1v_{2r+1}\}$. Clearly, M_1, M_2, M_3 are three line k -forests, and hence $\text{la}_k(C_n) \leq 3$. One can easily check that $\text{la}_k(C_n) = 3$ for n is odd and $k = 1$. ■

Lemma 3.3 [10] *For complete graph K_n ($n \geq 2$), $\text{la}_1(K_n) = \lceil n/2 \rceil$.*

Lemma 3.4 *For complete graph K_n ($n \geq 2$), $\lceil n/2 \rceil \leq \text{la}_k(K_n) \leq n$.*

Proof. From Lemma 2.1, we have

$$\lceil n/2 \rceil = \text{la}_{n-1}(K_n) \leq \text{la}_k(K_n) \leq \text{la}_1(K_n) \leq \Delta(K_n) + 1 = n,$$

as desired.

The Peterson graph HP_3 are shown in Figure 3 (a). We now turn our attention to study the linear k -arboricity of Peterson graphs.

Lemma 3.5 For Peterson graph HP_3 , $\text{la}_1(HP_3) = 4$.

Proof. Since HP_3 contains a cycle C_5 as its subgraph, it follows that $\text{la}_1(HP_3) \geq \text{la}_1(C_5) = 3$. The forest F_1 induced by the edges in $\{v_1v_2, v_7v_{10}, v_6v_9, v_3v_4\}$, the forest F_2 induced by the edges in $\{v_1v_5, v_2v_3, v_8v_{10}\}$, the forest F_3 induced by the edges in $\{v_4v_5, v_7v_9, v_6v_8\}$ and the forest F_4 induced by the edges in $\{v_1v_6, v_2v_7, v_3v_8, v_4v_9, v_5v_{10}\}$ form 4 linear 1-forests in HP_3 . So $3 \leq \text{la}_k(HP_3) \leq 4$.

We claim that $\text{la}_k(HP_3) = 4$. Assume, to the contrary, that $\text{la}_k(HP_3) = 3$. Then HP_3 can be decomposed into 3 linear 1-forests, say F_1, F_2, F_3 . Set $C_1 = v_6v_8v_{10}v_7v_9v_6$, $C_2 = v_1v_2v_3v_4v_5v_1$, and $M = \{v_1v_6, v_2v_7, v_3v_8, v_4v_9, v_5v_{10}\}$. We distinguish the following cases to show this claim.

Case 1. $|M \cap E(F_1)| = 5$ and $|M \cap E(F_2)| = |M \cap E(F_3)| = 0$.

Observe that all the edges in C_1 does not belong to F_1 . Otherwise, there is a path induced by the edges in F_1 such that its length is at least 2, which contradicts to the fact that $k = 1$. So all the edges in C_1 must belong to F_2 or F_3 . Since C_1 is a cycle of order 5, there is a path of length at least 2 in F_2 or F_3 , also a contradiction.

Case 2. $|M \cap E(F_1)| = 4$, $|M \cap E(F_2)| = 1$ and $|M \cap E(F_3)| = 0$.

Without loss of generality, let $v_1v_6 \in F_2$ and $M \setminus \{v_1v_6\} \subseteq E(F_1)$. Note that all the edges in C_1 does not belong to F_1 . So all the edges in C_1 must belong to F_2 or F_3 . Since $k = 1$, it follows that the elements in $E(C_1)$ must belongs to at least 3 linear 1-forests, a contradiction.

Case 3. $|M \cap E(F_1)| = 3$ and $|M \cap E(F_2)| = 2$ and $|M \cap E(F_3)| = 0$, or $|M \cap E(F_1)| = 3$ and $|M \cap E(F_2)| = |M \cap E(F_3)| = 1$.

From the symmetry of HP_3 , we only need to consider the two cases $v_1v_6, v_2v_7 \notin E(F_1)$ and $v_1v_6, v_3v_8 \notin E(F_1)$. At first, we consider the former case and suppose $v_1v_6, v_2v_7 \notin E(F_1)$. Since $k = 1$, it follows that $E(F_1) \cap E(C_1) = \emptyset$. So all the edges in C_1 must belong to F_2 or F_3 . Since $k = 1$, it follows that the elements in $E(C_1)$ must belongs to at least 3 linear 1-forests, a contradiction. Next, we consider the latter case and suppose $v_1v_6, v_3v_8 \notin E(F_1)$. Since $k = 1$, it follows that $E(F_1) \cap E(C_2) = \emptyset$. So all the edges in C_2 must belong to F_2 or F_3 . Since $k = 1$, it follows that the elements in $E(C_2)$ must belongs to at least 3 linear 1-forests, a contradiction.

Case 4. $|M \cap E(F_1)| = 2$, $|M \cap E(F_2)| = 2$ and $|M \cap E(F_3)| = 1$.

From the symmetry of HP_3 , we have the following cases to consider:

- (1) $v_1v_6, v_2v_7 \in E(F_1)$, $v_3v_8, v_4v_9 \in E(F_2)$, $v_5v_{10} \in E(F_3)$;
- (2) $v_1v_6, v_2v_7 \in E(F_1)$, $v_3v_8, v_5v_{10} \in E(F_2)$, $v_4v_9 \in E(F_3)$;
- (3) $v_1v_6, v_3v_8 \in E(F_1)$, $v_2v_7, v_4v_9 \in E(F_2)$, $v_5v_{10} \in E(F_3)$;
- (4) $v_1v_6, v_3v_8 \in E(F_1)$, $v_2v_7, v_5v_{10} \in E(F_2)$, $v_4v_9 \in E(F_3)$.

For (1), since $v_1v_6, v_2v_7 \in E(F_1)$, it follows that all the edges adjacent to v_1v_6 and v_2v_7 does not belong to F_1 . Then the elements in $\{v_3v_8, v_4v_9, v_5v_{10}, v_3v_4, v_4v_5, v_8v_{10}\}$ can belong to F_1 . Note that $v_3v_8, v_4v_9 \in E(F_2)$, $v_5v_{10} \in E(F_3)$, and v_3v_4, v_4v_5 are adjacent. So $2 \leq |E(F_1)| \leq 4$. We have the following claim.

Claim 1. $|E(F_1)| = 4$.

Proof of Claim 1. Assume, to the contrary, that $|E(F_1)| = 2$ or $|E(F_1)| = 3$. Then $|E(F_1) \cap E(C_1)| = 0$ or $|E(F_1) \cap E(C_2)| = 0$. Without loss of generality, let $|E(F_1) \cap E(C_1)| = 0$. So all the edges in C_1 must belong to F_2 or F_3 . Since $k = 1$, it follows that the elements in $E(C_1)$ must belongs to at least 3 linear 1-forests, a contradiction. ■

From Claim 1, $|E(F_1)| = 4$. Then $F_1 = \{v_1v_6, v_2v_7, v_4v_5, v_8v_{10}\}$ or $F_1 = \{v_1v_6, v_2v_7, v_3v_4, v_8v_{10}\}$. Suppose $F_1 = \{v_1v_6, v_2v_7, v_4v_5, v_8v_{10}\}$. Note that $v_3v_8, v_4v_9 \in E(F_2)$, $v_5v_{10} \in E(F_3)$. Then the edges in $E(C_2) \setminus \{v_4v_5\}$ belong to F_2 or F_3 . Since the subgraph induced by these edges is a path of length 4, it follows that $v_5v_1, v_2v_3 \in E(F_2)$ or $v_1v_2, v_3v_4 \in E(F_2)$. Whenever which case happens, we have a path of length at least 2 in F_2 , a contradiction.

Similarly to the proof of (2), we can also prove the correctness of (2)-(4). ■

The following observation is immediate, which will be used in lemma 3.6.

Observation 3.1 *Let $C_5 = w_1w_2w_3w_4w_5w_1$ be a cycle. If $\text{la}_3(C_5) = 2$, then C_5 can be decomposed into two linear 3-forests F_1, F_2 such that $F_1 = w_1w_2w_3w_4, F_2 = w_4w_5w_1$, or $F_1 = w_1w_2w_3 \cup w_4w_5, F_2 = w_3w_4 \cup w_5w_1$.*

Lemma 3.6 *For Peterson graph HP_3 , $\text{la}_3(HP_3) = 3$.*

Proof. Note that the forest F_1 induced by the edges in $\{v_2v_1v_5v_4, v_9v_7v_{10}v_8\}$, the forest F_2 induced by the edges in $\{v_9v_6v_8, v_4v_3v_2\}$ and the forest F_3 induced by the edges in $\{v_1v_6, v_2v_7, v_3v_8, v_4v_9, v_5v_{10}\}$ form linear 3-forests in HP_3 . So $\text{la}_k(HP_3) \leq 3$.

It suffices to show that $\text{la}_3(HP_3) \geq 3$. Since HP_3 contains cycles, it follows that $2 \leq \text{la}_k(HP_3) \leq 3$. We claim that $\text{la}_3(HP_3) = 3$. Assume, to the contrary, that $\text{la}_3(HP_3) = 2$. Then HP_3 can be decomposed into 2 linear 3-forests, say F_1, F_2 . Set $C_1 = v_6v_8v_{10}v_7v_9v_6$, $C_2 = v_1v_2v_3v_4v_5v_1$, and $M = \{v_1v_6, v_2v_7, v_3v_8, v_4v_9, v_5v_{10}\}$. Note that C_1, C_2 are two cycles, $E(C_i) \cap E(F_1) \neq \emptyset$ and $E(C_i) \cap E(F_2) \neq \emptyset$ for $i = 1, 2$. Since $k = 3$ and $\text{la}_3(HP_3) = 2$, it follows from Observation 3.1 that

- (1) $E(C_1) \cap E(F_1) = \{v_6v_8, v_8v_{10}, v_{10}v_7\}$ and $E(C_1) \cap E(F_2) = \{v_7v_9, v_6v_9\}$.
- (2) $E(C_1) \cap E(F_1) = \{v_7v_9, v_6v_9\}$ and $E(C_1) \cap E(F_2) = \{v_6v_8, v_8v_{10}, v_{10}v_7\}$.
- (3) $E(C_1) \cap E(F_1) = \{v_6v_8, v_6v_9, v_7v_{10}\}$ and $E(C_1) \cap E(F_2) = \{v_7v_9, v_8v_{10}\}$.
- (4) $E(C_1) \cap E(F_1) = \{v_7v_9, v_8v_{10}\}$ and $E(C_1) \cap E(F_2) = \{v_6v_8, v_6v_9, v_7v_{10}\}$.

By symmetry, we only need to consider (1) and (3). For (1), we claim that there is at most one edge in M belonging to F_1 . Otherwise, there exists a vertex of degree 3 in F_1 or a path of length at least 4, a contradiction. So there is at most one edge in M belonging to F_1 . Furthermore, there are at least four edges in M belonging to F_2 . Then there exists a path of length at least 4 in F_2 , also a contradiction.

We conclude that (3) holds. Similarly, C_2 must have the same decomposition as C_1 . By symmetry of HP_3 , we have the following.

$$(3.1) \quad E(C_1) \cap E(F_1) = \{v_6v_8, v_6v_9, v_7v_{10}\}, E(C_1) \cap E(F_2) = \{v_7v_9, v_8v_{10}\}, E(C_2) \cap E(F_1) = \{v_1v_5, v_2v_3, v_3v_4\}, E(C_2) \cap E(F_2) = \{v_4v_5, v_1v_2\}.$$

$$(3.2) \quad E(C_1) \cap E(F_1) = \{v_6v_8, v_6v_9, v_7v_{10}\}, E(C_1) \cap E(F_2) = \{v_7v_9, v_8v_{10}\}, E(C_2) \cap E(F_1) = \{v_1v_2, v_4v_5\}, E(C_2) \cap E(F_2) = \{v_1v_5, v_2v_3, v_3v_4\}.$$

$$(3.3) \quad E(C_1) \cap E(F_1) = \{v_6v_8, v_6v_9, v_7v_{10}\}, E(C_1) \cap E(F_2) = \{v_7v_9, v_8v_{10}\}, E(C_2) \cap E(F_1) = \{v_1v_2, v_1v_5, v_3v_4\}, E(C_2) \cap E(F_2) = \{v_4v_5, v_2v_3\}.$$

$$(3.4) \quad E(C_1) \cap E(F_1) = \{v_6v_8, v_6v_9, v_7v_{10}\}, E(C_1) \cap E(F_2) = \{v_7v_9, v_8v_{10}\}, E(C_2) \cap E(F_1) = \{v_2v_3, v_4v_5\}, E(C_2) \cap E(F_2) = \{v_1v_2, v_1v_5, v_3v_4\}.$$

$$(3.5) \quad E(C_1) \cap E(F_1) = \{v_6v_8, v_6v_9, v_7v_{10}\}, E(C_1) \cap E(F_2) = \{v_7v_9, v_8v_{10}\}, E(C_2) \cap E(F_1) = \{v_1v_2, v_2v_3, v_4v_5\}, E(C_2) \cap E(F_2) = \{v_1v_5, v_3v_4\}.$$

$$(3.6) \quad E(C_1) \cap E(F_1) = \{v_6v_8, v_6v_9, v_7v_{10}\}, E(C_1) \cap E(F_2) = \{v_7v_9, v_8v_{10}\}, E(C_2) \cap E(F_1) = \{v_1v_5, v_3v_4\}, E(C_2) \cap E(F_2) = \{v_1v_2, v_2v_3, v_4v_5\}.$$

We only prove that (3.1) is not true, and the other five cases can be discussed similarly. For (3.1), we claim that $v_2v_7 \in E(F_2)$. Otherwise, the path induced by the edges in $\{v_4v_3, v_3v_2, v_2v_7, v_7v_{10}\}$ has length 4, a contradiction. So $v_2v_7 \in E(F_2)$. We now focus our attention to the edge v_4v_9 . If $v_4v_9 \in E(F_1)$, then the path induced by the edges in $\{v_2v_3, v_3v_4, v_4v_9, v_9v_6, v_6v_8\}$ has length 5, a contradiction. If $v_4v_9 \in E(F_2)$, then the path induced by the edges in $\{v_1v_2, v_2v_7, v_7v_9, v_9v_4, v_4v_5\}$ has length 5, a contradiction. ■

From Claim 1, we have $\text{la}_k(HP_3) = 3$, as desired.

Proposition 3.2 *For a Peterson graph HP_3 ,*

$$\text{la}_k(HP_3) = \begin{cases} 4 & \text{if } k = 1, \\ 3 & \text{if } k = 2, \\ 3 & \text{if } k = 3, \\ 2 & \text{if } k \geq 4. \end{cases}$$

Proof. From Lemmas 3.4 and 3.5, the results follow for $k = 1, 3$. For $k = 2$, the forest F_1 induced by the paths in $\{v_1v_5v_{10}, v_6v_9v_7, v_2v_3v_8\}$, the forest F_2 induced by the paths

in $\{v_3v_4v_9, v_6v_8v_{10}, v_1v_2v_7\}$, and the forest F_3 induced by the edges in $\{v_1v_6, v_7v_{10}, v_4v_5\}$ form linear 2-forests in HP_3 . So $\text{la}_2(HP_3) \leq 3$. From Lemma 2.1 and Lemma 3.5, we have $\text{la}_2(HP_3) \geq \text{la}_3(HP_3) = 3$ and hence $\text{la}_2(HP_3) = 3$, as desired. For $k \geq 4$, the forest

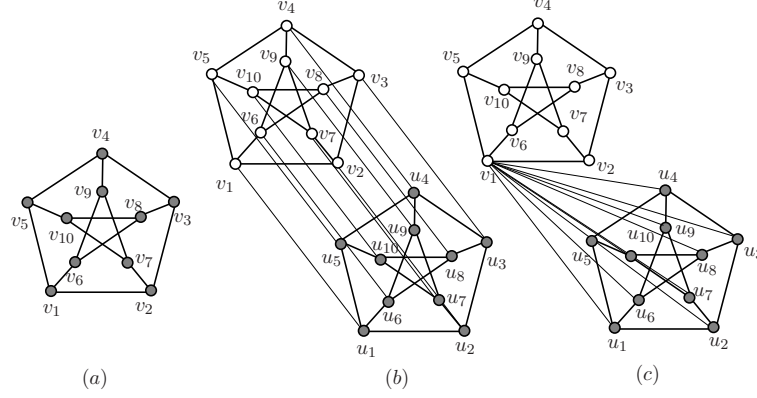


Figure 3: (a) Petersen graph; (b) The network HP_4 ; (c) The structure of HL_4 .

F_1 induced by the paths in $\{v_3v_2v_1v_5v_4, v_7v_9v_6v_8v_{10}\}$ and the forest F_2 induced by the paths in $\{v_8v_3v_4v_9, v_5v_{10}v_7v_2, v_1v_6\}$ form linear k -forests in HP_3 . So $\text{la}_k(HP_3) \leq 2$. From Observation 2.4, we have $\text{la}_k(HP_3) = 2$. ■

3.1 Two-dimensional grid graph

A *two-dimensional grid graph* is an $m \times n$ graph $G_{n,m}$ that is the graph Cartesian product $P_n \square P_m$ of path graphs on m and n vertices. For more details on grid graph, we refer to [7, 18]. The network $P_n \circ P_m$ is the graph lexicographical product $P_n \circ P_m$ of path graphs on m and n vertices. For more details on $P_n \circ P_m$, we refer to [24].

Proposition 3.3 (i) For network $P_n \square P_m$ ($m \geq n \geq 3$),

$$\begin{cases} \text{la}_k(P_n \square P_m) = 2 & \text{if } k \geq \max\{m-1, n-1\}, \\ 2 \leq \text{la}_k(P_n \square P_m) \leq 3 & \text{if } k \geq m-1, k \leq n-1, \\ 2 \leq \text{la}_k(P_n \square P_m) \leq 3 & \text{if } k \geq n-1, k \leq m-1, \\ 2 \leq \text{la}_k(P_n \square P_m) \leq 4 & \text{if } k \leq \max\{m-1, n-1\}. \end{cases}$$

(ii) For network $P_n \circ P_m$ ($n \geq 4, m \geq 3$),

$$\begin{cases} \text{la}_k(P_n \circ P_m) = m+1 & \text{if } k \geq \max\{m-1, n-1\}, \\ m+1 \leq \text{la}_k(P_n \circ P_m) \leq 2m+1 & \text{if } k \geq m-1, k \leq n-1, \\ m+1 \leq \text{la}_k(P_n \circ P_m) \leq m+2 & \text{if } k \geq n-1, k \leq m-1, \\ m+1 \leq \text{la}_k(P_n \circ P_m) \leq 2m+2 & \text{if } k \leq \max\{m-1, n-1\}. \end{cases}$$

(iii) For network $P_n \times P_m$ ($n \geq 4, m \geq 3$),

$$\begin{cases} \text{la}_k(P_n \times P_m) = 2 & \text{if } k \geq \max\{m-1, n-1\}, \\ 2 \leq \text{la}_k(P_n \times P_m) \leq 4 & \text{if } k \geq m-1, k \leq n-1, \\ 2 \leq \text{la}_k(P_n \times P_m) \leq 4 & \text{if } k \geq n-1, k \leq m-1, \\ 2 \leq \text{la}_k(P_n \times P_m) \leq 8 & \text{if } k \leq \max\{m-1, n-1\}. \end{cases}$$

(iv) For network $P_n \boxtimes P_m$ ($n \geq 4, m \geq 3$),

$$\begin{cases} \text{la}_k(P_n \boxtimes P_m) = 4 & \text{if } k \geq \max\{m-1, n-1\}, \\ 4 \leq \text{la}_k(P_n \boxtimes P_m) \leq 7 & \text{if } k \geq m-1, k \leq n-1, \\ 4 \leq \text{la}_k(P_n \boxtimes P_m) \leq 7 & \text{if } k \geq n-1, k \leq m-1, \\ 4 \leq \text{la}_k(P_n \boxtimes P_m) \leq 12 & \text{if } k \leq \max\{m-1, n-1\}. \end{cases}$$

Proof. (i) From Observation 2.4, $\text{la}_k(P_n \square P_m) \geq 2$. From Theorem 2.5, we have

$$\text{la}_k(P_n \square P_m) \leq \text{la}_k(P_n) + \text{la}_k(P_m) \leq \begin{cases} 2 & \text{if } k \geq \max\{m-1, n-1\}, \\ 3 & \text{if } k \geq m-1, k \leq n-1, \\ 3 & \text{if } k \geq n-1, k \leq m-1, \\ 4 & \text{if } k \leq \max\{m-1, n-1\}. \end{cases}$$

The result follows.

(ii) From Theorem 2.9, we have $\text{la}_k(P_n \circ P_m) \geq \left\lceil \frac{\Delta(P_m) + |V(P_m)|\Delta(P_n)}{2} \right\rceil = m+1$. From Theorem 2.9, we have

$$\text{la}_k(P_n \circ P_m) \leq \text{la}_k(P_n)|V(P_m)| + \text{la}_k(P_m) = \begin{cases} m+1 & \text{if } k \geq \max\{m-1, n-1\}, \\ 2m+1 & \text{if } k \geq m-1, k \leq n-1, \\ m+2 & \text{if } k \geq n-1, k \leq m-1, \\ 2m+2 & \text{if } k \leq \max\{m-1, n-1\}. \end{cases}$$

(iii) From Theorem 2.10, we have $\text{la}_k(P_n \times P_m) \geq \left\lceil \frac{\Delta(P_m)\Delta(P_n)}{2} \right\rceil = 2$. From Theorem 2.10 and Lemma 3.1, we have

$$\text{la}_k(P_n \times P_m) \leq 2\text{la}_k(P_n)\text{la}_k(P_m) = \begin{cases} 2 & \text{if } k \geq \max\{m-1, n-1\}, \\ 4 & \text{if } k \geq m-1, k \leq n-1, \\ 4 & \text{if } k \geq n-1, k \leq m-1, \\ 8 & \text{if } k \leq \max\{m-1, n-1\}. \end{cases}$$

(iv) From Observation 1.1, (i) and (iii) of this proposition, we have $\text{la}_k(P_n \boxtimes P_m) \leq \text{la}_k(P_n \square P_m) + \text{la}_k(P_n \times P_m)$ and hence

$$\text{la}_k(P_n \boxtimes P_m) \leq \begin{cases} 4 & \text{if } k \geq \max\{m-1, n-1\}, \\ 7 & \text{if } k \geq m-1, k \leq n-1, \\ 7 & \text{if } k \geq n-1, k \leq m-1, \\ 12 & \text{if } k \leq \max\{m-1, n-1\}. \end{cases}$$

From Theorem 2.11, we have $\text{la}_k(P_n \boxtimes P_m) \geq \left\lceil \frac{\Delta(P_m)\Delta(P_n) + \Delta(P_m) + \Delta(P_n)}{2} \right\rceil = 4$. ■

Remark 1. Let $G = P_n$ and $H = P_m$. For $k \geq \max\{m-1, n-1\}$, we have $\text{la}_k(P_n \square P_m) = 2 = \text{la}_k(P_n) + \text{la}_k(P_m)$, which implies that the upper bound of Theorem 2.5 is sharp for $k = \ell$ and $k \geq \max\{m-1, n-1\}$. For $k \geq \max\{m-1, n-1\}$, from Theorem 2.9, we have $m+1 = \left\lceil \frac{\Delta(P_m) + |V(P_m)|\Delta(P_n)}{2} \right\rceil \leq \text{la}_k(P_n \circ P_m) \leq \text{la}_k(P_n)|V(P_m)| + \text{la}_k(P_m) = m+1$, which implies that the upper and lower bounds of Theorem 2.9 are sharp for $k = \ell$ and $k \geq \max\{m-1, n-1\}$. For $k \geq \max\{m-1, n-1\}$, from Theorem 2.10, we have $2 = \left\lceil \frac{\Delta(P_n)\Delta(P_m)}{2} \right\rceil \leq \text{la}_k(P_n \times P_m) \leq 2\text{la}_k(P_n)\text{la}_k(P_m) = 2$, which implies that the upper and lower bounds of Theorem 2.10 are sharp for $k = \ell$ and $k \geq \max\{m-1, n-1\}$. For $k \geq \max\{m-1, n-1\}$, from Theorem 2.11, we have $4 = \left\lceil \frac{\Delta(P_n)\Delta(P_m) + \Delta(P_n) + \Delta(P_m)}{2} \right\rceil \leq \text{la}_k(P_n \boxtimes P_m) \leq \text{la}_k(P_n) + \text{la}_k(P_m) + 2\text{la}_k(P_n)\text{la}_k(P_m) = 4$, which implies that the upper and lower bounds of Theorem 2.11 are sharp for $k = \ell$ and $k \geq \max\{m-1, n-1\}$.

3.2 r -dimensional mesh

An r -dimensional mesh is the Cartesian product of r paths. By this definition, two-dimensional grid graph is a 2-dimensional mesh. An r -dimensional hypercube is a special case of an r -dimensional mesh, in which the r paths are all of size 2; see [20].

Proposition 3.4 (i) For r -dimensional mesh $P_{m_1} \square P_{m_2} \square \cdots \square P_{m_r}$,

$$2 \leq \text{la}_k(P_{m_1} \square P_{m_2} \square \cdots \square P_{m_r}) \leq 2r.$$

(ii) For network $P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_r}$,

$$1 + \prod_{i=2}^r m_i m_{i+1} \cdots m_r \leq \text{la}_k(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_r}) \leq 2 \left(\sum_{i=2}^r m_i m_{i+1} \cdots m_r + 1 \right).$$

(iii) For network $P_{m_1} \times P_{m_2} \times \cdots \times P_{m_r}$,

$$2^{r-1} \leq \text{la}_k(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_r}) \leq 2^{2r-1}.$$

(iv) For network $P_{m_1} \boxtimes P_{m_2} \boxtimes \cdots \boxtimes P_{m_r}$,

$$\frac{1}{2}(3^r - 1) \leq \text{la}_k(P_{m_1} \boxtimes P_{m_2} \boxtimes \cdots \boxtimes P_{m_r}) \leq 2r + 2^{2r-1}.$$

Proof. (i) From Corollary 2.6 and Lemma 3.1, we have $\text{la}_k(P_{m_1} \square P_{m_2} \square \cdots \square P_{m_r}) \leq \text{la}_k(P_{m_1}) + \text{la}_k(P_{m_2}) + \cdots + \text{la}_k(P_{m_r}) \leq 2r$. From this together with Observation 2.4, we have $2 \leq \text{la}_k(P_{m_1}) + \text{la}_k(P_{m_2}) + \cdots + \text{la}_k(P_{m_r}) \leq 2r$, as desired.

(ii) From Theorem 2.9, we have $\text{la}_k(G \circ H) \leq \text{la}_k(G)|V(H)| + \text{la}_k(H)$ for any two graphs G and H , and hence

$$\begin{aligned}
& \text{la}_k(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_r}) \\
&= \text{la}_k((P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-1}}) \circ P_{m_r}) \\
&\leq \text{la}_k(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-1}})m_r + \text{la}_k(P_{m_r}) \\
&\leq \text{la}_k(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-1}})m_r + 2 \\
&\leq [\text{la}_k(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-2}})m_{r-1} + \text{la}_k(P_{m_{r-1}})]m_r + 2 \\
&\leq [\text{la}_k(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-2}})m_{r-1} + 2]m_r + 2 \\
&= \text{la}_k(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-2}})m_{r-1}m_r + 2m_r + 2 \\
&\leq [\text{la}_k(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-3}})m_{r-2} + \text{la}_k(P_{m_{r-2}})]m_{r-1}m_r + 2m_r + 2 \\
&\leq \text{la}_k(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-3}})m_{r-2}m_{r-1}m_r + 2m_{r-1}m_r + 2m_r + 2 \\
&\leq \dots \\
&\leq \text{la}_k(P_{m_1} \circ P_{m_2})m_3 \dots m_{r-2}m_{r-1}m_r + 2m_4 \dots m_{r-2}m_{r-1}m_r + \dots + 2m_{r-1}m_r + 2m_r + 2 \\
&\leq 2m_2m_3 \dots m_{r-2}m_{r-1}m_r + 2m_3 \dots m_{r-2}m_{r-1}m_r + \dots + 2m_{r-1}m_r + 2m_r + 2 \\
&\leq 2 \sum_{i=2}^r m_i m_{i+1} \dots m_r + 2.
\end{aligned}$$

From Theorem 2.9, we have $\text{la}_k(G \circ H) \geq \left\lceil \frac{\Delta(H) + |V(H)|\Delta(G)}{2} \right\rceil$ for any two graphs G and H , and hence

$$\begin{aligned}
& \text{la}_k(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_r}) \\
&\geq \left\lceil \frac{\Delta(P_{m_r}) + |V(P_{m_r})|\Delta(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-1}})}{2} \right\rceil \\
&\geq 1 + \left\lceil \frac{m_r}{2} \Delta(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-1}}) \right\rceil \\
&\geq 1 + \left\lceil \frac{m_r}{2} (\Delta(P_{m_{r-1}}) + |V(P_{m_{r-1}})|\Delta(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-2}})) \right\rceil \\
&\geq 1 + m_r + \left\lceil \frac{m_r m_{r-1}}{2} \Delta(P_{m_1} \circ P_{m_2} \circ \cdots \circ P_{m_{r-2}}) \right\rceil \\
&\geq \dots \\
&\geq 1 + m_r + m_r m_{r-1} + \dots + \left\lceil \frac{m_r m_{r-1} \cdots m_2}{2} \Delta(P_{m_1}) \right\rceil \\
&\geq 1 + m_r + m_r m_{r-1} + \dots + m_r m_{r-1} \cdots m_2 \\
&\geq 1 + \prod_{i=2}^r m_i m_{i+1} \dots m_r.
\end{aligned}$$

(iii) From Theorem 2.10, we have $\text{la}_k(G \times H) \leq 2\text{la}_k(G)\text{la}_k(H)$ for any two graphs G and H , and hence

$$\begin{aligned}
\text{la}_k(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_r}) &\leq 2\text{la}_k(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_{r-1}})\text{la}_k(P_{m_r}) \\
&\leq 2^2\text{la}_k(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_{r-1}}) \\
&\leq 2^3\text{la}_k(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_{r-2}})\text{la}_k(P_{m_{r-1}}) \\
&\leq 2^4\text{la}_k(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_{r-2}}) \\
&\leq \dots \\
&\leq 2^{2(r-1)}\text{la}_k(P_{m_1}) \\
&\leq 2^{2r-1}.
\end{aligned}$$

From Theorem 2.10, we have $\text{la}_k(G \times H) \geq \left\lceil \frac{\Delta(G)\Delta(H)}{2} \right\rceil$ for any two graphs G and H , and hence

$$\begin{aligned}
\text{la}_k(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_r}) &\geq \left\lceil \frac{\Delta(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_{r-1}})\Delta(P_{m_r})}{2} \right\rceil \\
&\geq \Delta(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_{r-1}}) \\
&\geq \Delta(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_{r-2}})\Delta(P_{m_{r-1}}) \\
&= 2\Delta(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_{r-2}}) \\
&\geq 2\Delta(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_{r-3}})\Delta(P_{m_{r-2}}) \\
&= 2^2\Delta(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_{r-3}}) \\
&\geq \dots \\
&\geq 2^{r-2}\text{la}_k(P_{m_1}) \\
&\geq 2^{r-1}.
\end{aligned}$$

(iv) From (i), (iii) of this proposition and Observation 1.1, we have

$$\begin{aligned}
\text{la}_k(P_{m_1} \boxtimes P_{m_2} \boxtimes \cdots \boxtimes P_{m_r}) &\leq \text{la}_k(P_{m_1} \square P_{m_2} \square \cdots \square P_{m_r}) + \text{la}_k(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_r}) \\
&\leq 2r + 2^{2r-1}.
\end{aligned}$$

From Theorem 2.10, we have

$$\text{la}_k(G \boxtimes H) \geq \left\lceil \frac{\Delta(G)\Delta(H) + \Delta(G) + \Delta(H)}{2} \right\rceil = \left\lceil \frac{\Delta(G)(\Delta(H) + 1) + \Delta(H)}{2} \right\rceil$$

for any two graphs G and H , and hence

$$\begin{aligned}
& \text{la}_k(P_{m_1} \boxtimes P_{m_2} \boxtimes \cdots \boxtimes P_{m_r}) \\
& \geq \left\lceil \frac{\Delta(P_{m_1} \boxtimes P_{m_2} \boxtimes \cdots \boxtimes P_{m_{r-1}})(\Delta(P_{m_r}) + 1) + \Delta(P_{m_r})}{2} \right\rceil \\
& = \left\lceil \frac{3}{2} \Delta(P_{m_1} \boxtimes P_{m_2} \boxtimes \cdots \boxtimes P_{m_{r-1}}) \right\rceil + 1 \\
& \geq \left\lceil \frac{3}{2} \Delta(P_{m_1} \boxtimes P_{m_2} \boxtimes \cdots \boxtimes P_{m_{r-2}})(\Delta(P_{m_{r-1}}) + 1) + \Delta(P_{m_{r-1}}) \right\rceil + 1 \\
& = \left\lceil \frac{3^2}{2} \Delta(P_{m_1} \boxtimes P_{m_2} \boxtimes \cdots \boxtimes P_{m_{r-2}}) \right\rceil + 3 + 1 \\
& \geq \left\lceil \frac{3^2}{2} \Delta(P_{m_1} \boxtimes P_{m_2} \boxtimes \cdots \boxtimes P_{m_{r-3}})(\Delta(P_{m_{r-2}}) + 1) + \Delta(P_{m_{r-2}}) \right\rceil + 1 \\
& = \left\lceil \frac{3^3}{2} \Delta(P_{m_1} \boxtimes P_{m_2} \boxtimes \cdots \boxtimes P_{m_{r-3}}) \right\rceil + 3^2 + 3 + 1 \\
& \geq \dots \\
& \geq \left\lceil \frac{3^{r-1}}{2} \Delta(P_{m_1}) \right\rceil + 3^{r-2} + \dots + 3^2 + 3 + 1 \\
& = \frac{1}{2}(3^r - 1).
\end{aligned}$$

3.3 r -dimensional torus

An r -dimensional torus is the Cartesian product of r cycles $C_{m_1}, C_{m_2}, \dots, C_{m_r}$ of size at least three. The cycles C_{m_i} are not necessary to have the same size. Ku et al. [21] showed that there are r edge-disjoint spanning trees in an r -dimensional torus. The network $C_{m_1} \circ C_{m_2} \circ \cdots \circ C_{m_r}$ is investigated in [24]. Here, we consider the networks constructed by $C_{m_1} \square C_{m_2} \square \cdots \square C_{m_r}$ and $C_{m_1} \circ C_{m_2} \circ \cdots \circ C_{m_r}$, respectively.

Proposition 3.5 (i) For r -dimensional torus $C_{m_1} \square C_{m_2} \square \cdots \square C_{m_r}$,

$$2 \leq \text{la}_k(C_{m_1} \square C_{m_2} \square \cdots \square C_{m_r}) \leq 3r.$$

(ii) For network $C_{m_1} \circ C_{m_2} \circ \cdots \circ C_{m_r}$,

$$1 + \prod_{i=2}^r m_i m_{i+1} \dots m_r \leq \text{la}_k(C_{m_1} \circ C_{m_2} \circ \cdots \circ C_{m_r}) \leq 3 \left(\sum_{i=2}^r m_i m_{i+1} \dots m_r + 1 \right).$$

(ii) For network $C_{m_1} \times C_{m_2} \times \cdots \times C_{m_r}$,

$$2^{r-1} \leq \text{la}_k(C_{m_1} \times C_{m_2} \times \cdots \times C_{m_r}) \leq 3 \cdot 6^{r-1}.$$

(iv) For network $C_{m_1} \boxtimes C_{m_2} \boxtimes \cdots \boxtimes C_{m_r}$,

$$\frac{1}{2}(3^r - 1) \leq \text{la}_k(C_{m_1} \boxtimes C_{m_2} \boxtimes \cdots \boxtimes C_{m_r}) \leq 3(r + 6^{r-1}).$$

Proof. (i) From Corollary 2.6 and Lemma 3.4, we have

$$\begin{aligned}
2 &\leq \max\{\text{la}_k(C_{m_1}), \text{la}_k(C_{m_2}), \dots, \text{la}_k(C_{m_r})\} \\
&\leq \text{la}_k(C_{m_1} \square C_{m_2} \square \dots \square C_{m_r}) \\
&\leq \text{la}_k(C_{m_1}) + \text{la}_k(C_{m_2}) + \dots + \text{la}_k(C_{m_r}) \\
&\leq 3r.
\end{aligned}$$

(ii) From Theorem 2.5, we have $\text{la}_k(G \circ H) \leq \text{la}_k(G)|V(H)| + \text{la}_k(H)$ for any two graphs G and H , and hence

$$\begin{aligned}
&\text{la}_k(C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_r}) \\
&= \text{la}_k((C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_{r-1}}) \circ C_{m_r}) \\
&\leq \text{la}_k(C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_{r-1}})m_r + \text{la}_k(C_{m_r}) \\
&\leq \text{la}_k(C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_{r-1}})m_r + 3 \\
&\leq [\text{la}_k(C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_{r-2}})m_{r-1} + \text{la}_k(C_{m_{r-1}})]m_r + 3 \\
&\leq [\text{la}_k(C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_{r-2}})m_{r-1} + 3]m_r + 3 \\
&= \text{la}_k(C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_{r-2}})m_{r-1}m_r + 3m_r + 3 \\
&\leq [\text{la}_k(C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_{r-3}})m_{r-2} + \text{la}_k(C_{m_{r-2}})]m_{r-1}m_r + 3m_r + 3 \\
&\leq \text{la}_k(C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_{r-3}})m_{r-2}m_{r-1}m_r + 3m_{r-1}m_r + 3m_r + 3 \\
&\leq \dots \\
&\leq \text{la}_k(C_{m_1} \circ C_{m_2})m_3 \dots m_{r-2}m_{r-1}m_r + 3m_4 \dots m_{r-2}m_{r-1}m_r + \dots + 3m_{r-1}m_r + 3m_r + 3 \\
&\leq 3m_2m_3 \dots m_{r-2}m_{r-1}m_r + 3m_3 \dots m_{r-2}m_{r-1}m_r + \dots + 3m_{r-1}m_r + 3m_r + 3 \\
&\leq 3 \sum_{i=2}^r m_i m_{i+1} \dots m_r + 3.
\end{aligned}$$

From Observation 2.1 and (ii) of Proposition 3.4, we have $\text{la}_k(C_{m_1} \circ C_{m_2} \circ \dots \circ C_{m_r}) \geq \text{la}_k(P_{m_1} \circ P_{m_2} \circ \dots \circ P_{m_r}) \geq 1 + \prod_{i=2}^r m_i m_{i+1} \dots m_r$.

(iii) From Theorem 2.10, we have $\text{la}_k(G \times H) \leq 2\text{la}_k(G)\text{la}_k(H)$ for any two graphs G and H , and hence

$$\begin{aligned}
\text{la}_k(C_{m_1} \times C_{m_2} \times \dots \times C_{m_r}) &\leq 2\text{la}_k(C_{m_1} \times C_{m_2} \times \dots \times C_{m_{r-1}})\text{la}_k(C_{m_r}) \\
&\leq 6\text{la}_k(C_{m_1} \times C_{m_2} \times \dots \times C_{m_{r-1}}) \\
&\leq 6[2\text{la}_k(C_{m_1} \times C_{m_2} \times \dots \times C_{m_{r-2}})\text{la}_k(C_{m_{r-1}})] \\
&\leq 6^2\text{la}_k(C_{m_1} \times C_{m_2} \times \dots \times C_{m_{r-2}}) \\
&\leq \dots \\
&\leq 6^{r-1}\text{la}_k(C_{m_1}) \\
&\leq 3 \cdot 6^{r-1}.
\end{aligned}$$

From Observation 2.1 and (iii) of Proposition 3.4, we have $\text{la}_k(C_{m_1} \times C_{m_2} \times \cdots \times C_{m_r}) \geq \text{la}_k(P_{m_1} \times P_{m_2} \times \cdots \times P_{m_r}) \geq 2^{r-1}$.

(iv) From (i), (iv) of this proposition and Observation 1.1, we have

$$\begin{aligned} & \text{la}_k(C_{m_1} \boxtimes C_{m_2} \boxtimes \cdots \boxtimes C_{m_r}) \\ & \leq \text{la}_k(C_{m_1} \square C_{m_2} \square \cdots \square C_{m_r}) + \text{la}_k(C_{m_1} \times C_{m_2} \times \cdots \times C_{m_r}) \\ & \leq 3(r + 6^{r-1}). \end{aligned}$$

From Observation 2.1 and (iv) of Proposition 3.4, we have $\text{la}_k(C_{m_1} \boxtimes C_{m_2} \boxtimes \cdots \boxtimes C_{m_r}) \geq \text{la}_k(P_{m_1} \boxtimes P_{m_2} \boxtimes \cdots \boxtimes P_{m_r}) \geq 1 + \prod_{i=2}^r m_i m_{i+1} \cdots m_r$.

3.4 r -dimensional generalized hypercube

Let K_m be a clique of m vertices, $m \geq 2$. An r -dimensional generalized hypercube [12, 14] is the Cartesian product of r cliques. We have the following:

Proposition 3.6 (i) For generalized hypercube $K_{m_1} \square K_{m_2} \square \cdots \square K_{m_r}$ ($m_i \geq 2$, $r \geq 2$, $1 \leq i \leq r$),

$$\max \left\{ \left\lceil \frac{m_i}{2} \right\rceil \mid 1 \leq i \leq r \right\} \leq \text{la}_k(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_r}) \leq \sum_{i=1}^r m_i.$$

(ii) For network $K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_r}$ ($m_i \geq 2$, $r \geq 2$, $1 \leq i \leq r$),

$$\left\lceil \frac{\sum_{i=1}^r m_i}{2} \right\rceil \leq \text{la}_k(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_r}) \leq \frac{\sum_{i=1}^r m_i}{2}.$$

(iii) For network $K_{m_1} \times K_{m_2} \times \cdots \times K_{m_r}$ ($m_i \geq 2$, $r \geq 2$, $1 \leq i \leq r$),

$$\left\lceil \frac{1}{2} \prod_{i=1}^r (m_i - 1) \right\rceil \leq \text{la}_k(K_{m_1} \times K_{m_2} \times \cdots \times K_{m_r}) \leq 2^{r-1} \prod_{i=1}^r m_i.$$

(iv) For network $K_{m_1} \boxtimes K_{m_2} \boxtimes \cdots \boxtimes K_{m_r}$ ($m_i \geq 2$, $r \geq 2$, $1 \leq i \leq r$),

$$\left\lceil \frac{1}{2} \prod_{i=1}^r m_r m_{r-1} \cdots m_{i+1} (m_i - 1) \right\rceil \leq \text{la}_k(K_{m_1} \boxtimes K_{m_2} \boxtimes \cdots \boxtimes K_{m_r}) \leq \sum_{i=1}^r m_i + 2^{r-1} \prod_{i=1}^r m_i.$$

Proof. (i) From Corollary 2.6 and Lemma 3.4, we have

$$\begin{aligned} \max \left\{ \left\lceil \frac{m_i}{2} \right\rceil \mid 1 \leq i \leq r \right\} &= \max \{ \text{la}_k(K_{m_1}), \text{la}_k(K_{m_2}), \dots, \text{la}_k(K_{m_r}) \} \\ &\leq \text{la}_k(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_r}) \\ &\leq \text{la}_k(K_{m_1}) + \text{la}_k(K_{m_2}) + \cdots + \text{la}_k(K_{m_r}) \\ &\leq \sum_{i=1}^r m_i. \end{aligned}$$

(ii) From the definition of lexicographical product, $K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_r}$ is a complete graph. From Lemma 3.4, we have

$$\left\lceil \frac{\sum_{i=1}^r m_i}{2} \right\rceil \leq \text{la}_k(K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_r}) \leq \frac{\sum_{i=1}^r m_i}{2}.$$

(iii) From Theorem 2.10, we have $\text{la}_k(G \times H) \leq 2\text{la}_k(G)\text{la}_k(H)$ for any two graphs G and H , and hence

$$\begin{aligned} \text{la}_k(K_{m_1} \times K_{m_2} \times \cdots \times K_{m_r}) &\leq 2\text{la}_k(K_{m_1} \times K_{m_2} \times \cdots \times K_{m_{r-1}})\text{la}_k(K_{m_r}) \\ &\leq 2m_r\text{la}_k(K_{m_1} \times K_{m_2} \times \cdots \times K_{m_{r-1}}) \\ &\leq 2^2m_r\text{la}_k(K_{m_1} \times K_{m_2} \times \cdots \times K_{m_{r-2}})\text{la}_k(K_{m_{r-1}}) \\ &\leq 2^2m_rm_{r-1}\text{la}_k(K_{m_1} \times K_{m_2} \times \cdots \times K_{m_{r-2}}) \\ &\leq \dots \\ &\leq 2^{r-1}m_rm_{r-1} \dots m_2\text{la}_k(K_{m_1}) \\ &= 2^{r-1} \prod_{i=1}^r m_i. \end{aligned}$$

From Theorem 2.10, we have $\text{la}_k(G \times H) \geq \left\lceil \frac{\Delta(G)\Delta(H)}{2} \right\rceil$ for any two graphs G and H , and hence

$$\begin{aligned} \text{la}_k(K_{m_1} \times K_{m_2} \times \cdots \times K_{m_r}) &\geq \left\lceil \frac{\Delta(K_{m_1} \times K_{m_2} \times \cdots \times K_{m_{r-1}})\Delta(K_{m_r})}{2} \right\rceil \\ &\geq \left\lceil \frac{m_r - 1}{2} \Delta(K_{m_1} \times K_{m_2} \times \cdots \times K_{m_{r-1}}) \right\rceil \\ &\geq \left\lceil \frac{m_r - 1}{2} \Delta(K_{m_1} \times K_{m_2} \times \cdots \times K_{m_{r-2}})\Delta(K_{m_r}) \right\rceil \\ &\geq \left\lceil \frac{(m_r - 1)(m_{r-1} - 1)}{2} \Delta(K_{m_1} \times K_{m_2} \times \cdots \times K_{m_{r-2}}) \right\rceil \\ &\geq \dots \\ &\geq \left\lceil \frac{(m_r - 1)(m_{r-1} - 1) \dots (m_2 - 1)}{2} \Delta(K_{m_1}) \right\rceil \\ &= \left\lceil \frac{1}{2} \prod_{i=1}^r (m_i - 1) \right\rceil. \end{aligned}$$

(iv) From (i), (iii) of this proposition and Observation 1.1, we have

$$\begin{aligned} &\text{la}_k(K_{m_1} \boxtimes K_{m_2} \boxtimes \cdots \boxtimes K_{m_r}) \\ &\leq \text{la}_k(K_{m_1} \square K_{m_2} \square \cdots \square K_{m_r}) + \text{la}_k(K_{m_1} \times K_{m_2} \times \cdots \times K_{m_r}) \\ &\leq \sum_{i=1}^r m_i + 2^{r-1} \prod_{i=1}^r m_i. \end{aligned}$$

From Theorem 2.11, we have

$$\text{la}_k(G \boxtimes H) \geq \left\lceil \frac{\Delta(G)(\Delta(H) + 1) + \Delta(H)}{2} \right\rceil$$

for any two graphs G and H , and hence

$$\begin{aligned} & \text{la}_k(K_{m_1} \boxtimes K_{m_2} \boxtimes \cdots \boxtimes K_{m_r}) \\ \geq & \left\lceil \frac{\Delta(K_{m_1} \boxtimes K_{m_2} \boxtimes \cdots \boxtimes K_{m_{r-1}})(\Delta(K_{m_r}) + 1) + \Delta(K_{m_r})}{2} \right\rceil \\ = & \left\lceil \frac{m_r - 1}{2} + \frac{m_r}{2} \Delta(K_{m_1} \boxtimes K_{m_2} \boxtimes \cdots \boxtimes K_{m_{r-1}}) \right\rceil \\ \geq & \left\lceil \frac{m_r - 1}{2} + \frac{m_r}{2} [\Delta(K_{m_1} \boxtimes K_{m_2} \boxtimes \cdots \boxtimes K_{m_{r-2}})(\Delta(K_{m_{r-1}}) + 1) + \Delta(K_{m_{r-1}})] \right\rceil \\ = & \left\lceil \frac{m_r - 1}{2} + \frac{m_r(m_{r-1} - 1)}{2} + \frac{m_r m_{r-1}}{2} \Delta(K_{m_1} \boxtimes K_{m_2} \boxtimes \cdots \boxtimes K_{m_{r-2}}) \right\rceil \\ \geq & \cdots \\ \geq & \left\lceil \frac{m_r - 1}{2} + \frac{m_r(m_{r-1} - 1)}{2} + \frac{m_r m_{r-1}(m_{r-2} - 1)}{2} + \cdots + \frac{m_r m_{r-1} \cdots m_2}{2} \Delta(K_{m_1}) \right\rceil \\ = & \left\lceil \frac{1}{2} \prod_{i=1}^r m_r m_{r-1} \cdots m_{i+1} (m_i - 1) \right\rceil. \end{aligned}$$

■

3.5 n -dimensional hyper Petersen network

An n -dimensional hyper Petersen network HP_n is the product of the well-known Petersen graph and Q_{n-3} [11], where $n \geq 3$ and Q_{n-3} denotes an $(n-3)$ -dimensional hypercube. The cases $n = 3$ and 4 of hyper Petersen networks are depicted in Figure 2. Note that HP_3 is just the Petersen graph; see Figure 3 (a).

The network HL_n is the lexicographical product of the Petersen graph and Q_{n-3} , where $n \geq 3$ and Q_{n-3} denotes an $(n-3)$ -dimensional hypercube; see [24]. Note that HL_4 is a graph obtained from two copies of the Petersen graph by add one edge between one vertex in a copy of the Petersen graph and one vertex in another copy; see Figure 3 (c) for an example (We only show the edges $v_1 u_i$ ($1 \leq i \leq 10$)).

Similarly, the networks HD_n and HS_n are defined as the direct and strong product of the Petersen graph and Q_{n-3} , respectively, where $n \geq 3$ and Q_{n-3} denotes an $(n-3)$ -dimensional hypercube. Note that $HL_3 = HD_3 = HS_3$ is just the Petersen graph, and

Proposition 3.7 (i) For network HP_4 ,

$$\begin{cases} 4 \leq \text{la}_k(HP_4) \leq 5 & \text{if } k = 1, \\ 3 \leq \text{la}_k(HP_4) \leq 4 & \text{if } k = 2, \\ 3 \leq \text{la}_k(HP_4) \leq 4 & \text{if } k = 3, \\ 2 \leq \text{la}_k(HP_4) \leq 3 & \text{if } k \geq 4. \end{cases}$$

(ii) For network HL_4 ,

$$\begin{cases} 4 \leq \text{la}_k(HL_4) \leq 14 & \text{if } k = 1, \\ 4 \leq \text{la}_k(HL_4) \leq 13 & \text{if } k = 2, \\ 4 \leq \text{la}_k(HL_4) \leq 13 & \text{if } k = 3, \\ 4 \leq \text{la}_k(HL_4) = 12 & \text{if } k \geq 4. \end{cases}$$

(iii) For network HD_4 ,

$$\begin{cases} 2 \leq \text{la}_k(HD_4) \leq 8 & \text{if } k = 1, \\ 2 \leq \text{la}_k(HD_4) \leq 6 & \text{if } k = 2, \\ 2 \leq \text{la}_k(HD_4) \leq 6 & \text{if } k = 3, \\ 2 \leq \text{la}_k(HD_4) \leq 4 & \text{if } k \geq 4. \end{cases}$$

(iv) For network HS_4 ,

$$\begin{cases} 4 \leq \text{la}_k(HS_4) \leq 13 & \text{if } k = 1, \\ 4 \leq \text{la}_k(HS_4) \leq 10 & \text{if } k = 2, \\ 4 \leq \text{la}_k(HS_4) \leq 10 & \text{if } k = 3, \\ 4 \leq \text{la}_k(HS_4) \leq 7 & \text{if } k \geq 4. \end{cases}$$

Proof. (i) From Theorem 2.5, we have

$$\text{la}_k(HP_4) \geq \text{la}_k(HP_3) \geq \begin{cases} 4 & \text{if } k = 1, \\ 3 & \text{if } k = 2, \\ 3 & \text{if } k = 3, \\ 2 & \text{if } k \geq 4. \end{cases}$$

From Theorem 2.5, Lemmas 3.1 and 3.5, we have

$$\text{la}_k(HP_4) \leq \text{la}_k(HP_3) + \text{la}_k(P_2) = \text{la}_k(HP_3) + 1 \leq \begin{cases} 5 & \text{if } k = 1, \\ 4 & \text{if } k = 2, \\ 4 & \text{if } k = 3, \\ 3 & \text{if } k \geq 4. \end{cases}$$

(ii) From Theorem 2.9, Lemmas 3.1 and 3.5, we have

$$4 \leq \text{la}_k(HP_4) \leq \text{la}_k(HP_3)|V(P_2)| + \text{la}_k(P_2) = 2\text{la}_k(HP_3) + 1 \leq \begin{cases} 9 & \text{if } k = 1, \\ 7 & \text{if } k = 2, \\ 7 & \text{if } k = 3, \\ 5 & \text{if } k \geq 4. \end{cases}$$

(iii) From Theorem 2.10, Lemmas 3.1 and 3.5, we have

$$\text{la}_k(HD_4) \leq 2\text{la}_k(HP_3)\text{la}_k(P_2) = 2\text{la}_k(HP_3) \leq \begin{cases} 8 & \text{if } k = 1, \\ 6 & \text{if } k = 2, \\ 6 & \text{if } k = 3, \\ 4 & \text{if } k \geq 4. \end{cases}$$

(iv) From Theorem 2.11, $\text{la}_k(HS_4) \geq 4$. From Observation 1.1, we have

$$\text{la}_k(HS_4) \leq \text{la}_k(HP_4) + \text{la}_k(HD_4) \leq \begin{cases} 13 & \text{if } k = 1, \\ 10 & \text{if } k = 2, \\ 10 & \text{if } k = 3, \\ 7 & \text{if } k \geq 4. \end{cases}$$

References

- [1] F. Bao, Y. Igarashi, S.R. Öhring, *Reliable Broadcasting in Product Networks*, Discrete Applied Math. 83 (1998), 3-20.
- [2] R.E.L. Aldred, N.C. Wormald, *More on the linear k -arboricity of regular graphs*, Australasian J. Combin. 18(1998), 97–104.
- [3] N. Alon, *The linear arboricity of graphs*, Israel J. Math. 62(3)(1988), 311–325.
- [4] N. Alon, V.J. Teague, N.C. Wormald, *Linear arboricity and linear k -arboricity of regular graphs*, Graphs & Combin. 17(1)(2001), 11–16.
- [5] J.C. Bermond, J.L. Fouquet, M. Habib, B. Péroche, *On linear k -arboricity*, Discrete Math. 52(2-3)(1984), 123–132.
- [6] J.A. Bondy, U.S.R. Murty, *Graph Theory*, GTM 244, Springer, 2008.
- [7] N.J. Calkin, H.S. Wilf, *The number of independent sets in a grid graph*, SIAM J. Discrete Math. 11(1)(1998), 54–60.
- [8] G.J. Chang, *Algorithmic aspects of linear k -arboricity*, Taiwanese J. Math. 3(1)(1999), 73–81.
- [9] G.J. Chang, B.L. Chen, H.L. Fu, K.C. Huang, *Linear k -arboricities on trees*, Discrete Appl. Math. 103(1-3)(2000), 281–287.
- [10] B.L. Chen, K.C. Huang, *On the linear k -arboricity of K_n and $K_{n,n}$* , Discrete Math. 254(1-3)(2002), 51–61.

- [11] S.K. Das, S.R. Öhring, A.K. Banerjee, *Embeddings into hyper Petersen network: Yet another hypercube-like interconnection topology*, VLSI Design 2(4)(1995), 335–351.
- [12] K. Day, A.-E. Al-Ayyoub, *The cross product of interconnection networks*, IEEE Trans. Parallel and Distributed Systems 8(2)(1997), 109–118.
- [13] G. Dirac, *On Hamilton circuits and Hamilton paths*, Math. Ann. 197(1972), 57–70.
- [14] P. Fragopoulou, S.G. Akl, H. Meijer, *Optimal communication primitives on the generalized hypercube network*, IEEE Trans. Parallel and Distributed Computing 32(2)(1996), 173–187.
- [15] R. Hammack, W. Imrich, Sandi Klavžr, *Handbook of product graphs*, Second edition, CRC Press, 2011.
- [16] M. Habib, B. Peroche, *La k -arboricité linéaire des arbres*, Annales Polonici Mathematici 17(1983), 307–317.
- [17] F. Harary, *Covering and packing in graphs. I*, Annals of the New York Academy of Sciences 175(1970), 198–205.
- [18] A. Itai, M. Rodeh, *The multi-tree approach to reliability in distributed networks*, Inform. Comput. 79(1988), 43–59.
- [19] B. Jackson, N. C. Wormald, *On the linear k -arboricity of cubic graphs*, Discrete Math. 162(1-3)(1996), 293–297.
- [20] S.L. Johnsson, C.T. Ho, *Optimum broadcasting and personalized communication in hypercubes*, IEEE Trans. Computers 38(9)(1989), 1249–1268.
- [21] S. Ku, B. Wang, T. Hung, *Constructing edge-disjoint spanning trees in product networks*, Parallel and Distributed Systems, IEEE Transactions on parallel and disjointed systems, 14(3)(2003), 213–221.
- [22] R. Laskar, B. Auerbach, *On decomposition of r -partite graphs into edge-disjoint Hamilton circuits*, Discrete Math. 14(1976), 265–268.
- [23] K.W. Lih, L.D. Tong, W.F. Wang, *The linear 2-arboricity of planar graphs*, Graphs & Combin. 19(2)(2003), 241–248.
- [24] Y. Mao, *Path-connectivity of lexicographical product graphs*, Int. J. Comput. Math. 93(1)(2016), 27–39.
- [25] A. Vesel, J. Žerovnik, *On the linear k -arboricity of cubic graphs*, Inter. J. Comput. Math. 75(4)(2000), 431–444.

- [26] B. Xue, L. Zuo, *Linear $(n - 1)$ -arboricity of $K_{n(m)}$* , Discrete Appl. Math. 158(14)(2010), 1546–1550.
- [27] C.H. Yeh, *Linear k -arboricity of complete multipartite graphs*, Thesis for Doctor Degree, National Chiao Tung University, 2005.
- [28] C.H. Yen, H.L. Fu, *Linear 2-arboricity of the complete graph*, Taiwanese J. Math. 14(1)(2010), 273–286.
- [29] L. Zuo, S. He, B. Xue, *The linear $(n - 1)$ -arboricity of Cartesian product graphs*, Appl. Anal. Discr. Math. 9(1)(2015), 13–28.